SOLVING THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN THE THREE-DIMENSIONAL CASE WITH SPECIAL REFERENCE TO THE PROBLEM OF SHAPING OF INTENSE CHARGED-PARTICLE BEAMS

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The hydrodynamic theory of intense charged-particle beams is one of the branches of the mechanics of continuous media. However, in contrast to, say, hydrodynamics, the problem in this case cannot be regarded as completely solved once one has integrated the beam equations which include the Maxwell equations and the equations of a charge in the self-consistent field. Within the framework of the standard formulation (the inverse problem) the solution defines flow in a half-space bounded by an infinite emitting surface, whereas real beams have finite dimensions. The flow in a domain with bounded transverse coordinates can be determined by finding the electrostatic Laplacian field equivalent to the discarded portion of the stream which matches continuously the solution of the beam equations at the boundary. This problem is called the "shaping" or "electrostatic focusing" problem; its mathematical expression is the Cauchy problem for the Laplace equation. We consider three-dimensional shaping problems in the case of cylindrical and axially symmetric charge-filled domains. We also write out the solutions for the periodic focusing of a cylindrical beam of arbitrary cross section and for the shaping of an elliptical beam with periodic variation of the z-component of the velocity; the expression for the potential outside a cylindrical beam of arbitrary cross section with emission bounded by a space charge is given in closed form.

The problem of shaping of intense charged-particle beams in mathematical formulation reduces to the Cauchy problem for the Laplace equation. Two-dimensional configurations are investigated in [1, 2]. In [1] the problem is solved by separation of variables and the solution expressed as a contour integral in the complex plane of the parameter p; the author of [2] used an analytical continuation of the Laplace equation to construct an expression for the potential by the Riemann method.

In dealing with three-dimensional problems in [3] we obtained solutions in the coordinate system x^i attached to the beam surface $x^1 = 0$ in the form of a series in powers of x^1 with coefficients which depend on x^2 and x^3. This method is suitable for obtaining the solution sufficiently near the boundary (the suitability criterion is the magnitude of the discrepancy) in the case where the Cauchy conditions are given by regular functions. This excludes streams whose velocity vanishes somewhere for a nonzero current density, and especially flows originating at the emitting surface (the potential near this surface during emission bounded by a space charge varies as $z^{4/3}$).

There appears to have been a single attempt [4] to determine the shaping electrodes for a cylindrical beam with an elliptical cross section for which the Cauchy conditions correspond to flow in a plane diode: $\varphi = z^{4/3}$, $\partial\varphi / \partial n = 0$. However, the results and the proposed approximate method of calculating equipotential surfaces is based on the false assumption that the solution is independent of η (ξ, η, z are the elliptical cylindrical coordinates). Hence, only the formulation of the problem in [4] remains valid.

In the present paper we solve the three-dimensional Cauchy problem for cylindrical domains of arbitrary smooth cross section and for two types of axially symmetric domains: a domain bounded by a surface of revolution with an arbitrary law of growth of the circular cross section and a toroid of arbitrary cross section. Our procedure is based on the isolation of z in the cylindrical case and of ψ (the azimuth) in the axially symmetric case by Fourier transformation (for regular Cauchy conditions) or Laplace transformation (for irregular functions) followed by exact solution of the remaining two-dimensional problem by the Riemann method.

To illustrate our technique we obtain approximate and exact solutions for the periodic focusing of a cylindrical beam of arbitrary cross section (the approximate solution approximates the true distribution of the potential at the boundary with the aid of the cosine [5]). We also write out the expression for the potential in the case of an elliptical beam with periodic variation of the z-component of the velocity [6].

In the case of a cylindrical beam with an arbitrary cross section the Lipschitz-Hankel condition enables us to integrate along a contour in the p-plane and to express the solution for the shaping of a flow described by a $^3/_2$ law in closed form. We show that the angle of inclination of the shaping electrode with a zero potential does not depend on the shape of the boundary and constitutes 67°.5 as in the two-dimensional case. Knowledge of the exact solution for the boundary which can be specified parametrically in terms of analytic functions enables us to construct an approximate expression for a non-analytic parametric equation (e.g. for a square).

The solution of the shaping problem of a toroidal beam of arbitrary cross section in the case of flow along circular trajectories [7] is written out in terms of contour integrals in the p-plane.

The properties of special functions used in the study are discussed in [8, 9].

1. A cylindrical domain of arbitrary cross section. The problem consists in solving the Laplace equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \qquad (1.1)$$

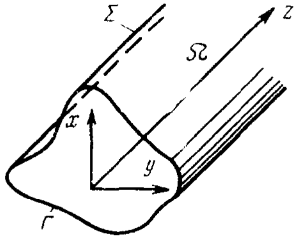

Fig. 1

when the Cauchy conditions are given at the boundary Σ of the domain Ω shown in Fig. 1. Let the directrix of the cylinder Γ be defined by the parametric equations

$$x = x_e(t), \qquad y = y_e(t)$$
$$dx_e / dt = \alpha(t), \qquad dy_e / dt = \beta(t) \qquad (1.2)$$

and let the potential and its normal derivative on Σ be given by the functions V_e and V_{ne}

$$\varphi|_\Sigma = V_e(t, z), \qquad \partial\varphi / \partial n|_\Sigma = V_{ne}(t, z) \qquad (1.3)$$

It is clear that the transformations

$$x + iy = x_e(w) + iy_e(w), \qquad w = u + iv$$

map the real axis $v = 0$ in the plane w onto Γ in the plane x, y. Under this mapping

Eq. (1.1) becomes

$$\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}+\sqrt{g}\,\frac{\partial^2\varphi}{\partial z^2}=0$$

where $g = g(u, v)$ is the determinant of the metric tensor g_{ik} of the system u, v, z.

If V_e, V_{ne} are regular in z, we shall attempt to find the solution of problem (1.1), (1.3) in the form

$$\varphi(u, v, z)=\int_{-\infty}^{\infty}\Phi(u, v; p)\,e^{ipz}\,dp$$

The function $\Phi(u, v; p)$, which we shall refer to as the "two-dimensional potential", then satisfies the equation

$$\frac{\partial^2\Phi}{\partial u^2}+\frac{\partial^2\Phi}{\partial v^2}-p^2\sqrt{g}\,\Phi=0 \qquad (1.4)$$

Here we have

$$\Phi|_{v=0}=\vartheta(u; p), \qquad \partial\Phi/\partial v|_{v=0}=f(u; p) \qquad (1.5)$$

at $v = 0$.

Here $\vartheta(u; p)$, $f(u; p)$ are the spectral densities of the Cauchy conditions,

$$\varphi|_{v=0}=V_e(u, z)=V(u, z), \quad \partial\varphi/\partial v|_{v=0}=(\alpha^2+\beta^2)^{1/2}V_{ne}(u, z)=F(u, z)$$

If we attempt to find the solution in the form of a Fourier series or with the aid of Laplace transformation, then these quantities are Fourier coefficients or the images of the functions V, F. In the latter case the p in (1.4) must be replaced by ip.

1.1. Solution of two-dimensional problem (1.4), (1.5). Let us effect the analytical continuation of the parameter u, replacing it by $u + i\xi$. Equation (1.7) then becomes hyperbolic,

$$\frac{\partial^2\Phi}{\partial v^2}-\frac{\partial^2\Phi}{\partial \xi^2}-p^2\sqrt{g}\,\Phi=0 \qquad (1.6)$$

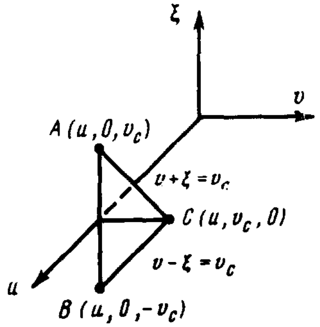

Fig. 2

The system u, v, ξ is shown in Fig. 2. Here C is the point at which the potential is calculated (the observation point); $v \pm \xi = v_c$ are the characteristics passing though this point. The Cauchy conditions are specified on AB as the analytical continuation of functions (1.5). Applying the Riemann method [10], we obtain

$$\Phi_C=\mathrm{Re}\,[\Phi_A+\int_0^{v_c}\left(G\frac{\partial\Phi}{\partial v}-\Phi\frac{\partial G}{\partial v}\right)d\xi] \qquad (1.7)$$

Here G (the Riemann function) satisfies the same equation as Φ and assumes the value unity at the characteristics, since the first derivatives in (1.6) are equal to zero,

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}-p^2G=0 \qquad (1.8)$$

The coordinates of the point C in the system x, y are defined by the expressions

$$x_c=\mathrm{Re}\,x_e(u+iv_c)-\mathrm{Im}\,y_e(u+iv_c)$$
$$y_c=\mathrm{Im}\,x_e(u+iv_c)+\mathrm{Re}\,y_e(u+iv_c)$$

Noting, in addition, that

$$(x-x_c)^2+(y-y_c)^2=0$$

on the characteristics $x \pm iy = x_c \pm iy_c$, we seek G in the form $G = G(\lambda)$, where

$$\lambda = ip[(x-x_c)^2+(y-y_c)^2]^{1/2}$$

It is easy to show that (1.8) then reduces to the Bessel equation

$$\lambda G'' + G' + \lambda G = 0$$

where the condition $G(0) = 1$ is satisfied by the zero-order Bessel function

$$G = J_0(\lambda) \tag{1.9}$$

Substituting (1.9) into (1.7), we arrive at the following final expression for $\Phi(u, v; p)$:

$$\Phi(u, v; p) = \operatorname{Re}\Big\{\vartheta(w; p) + \int_0^v [J_0(\lambda_e) f(\zeta; p) +$$

$$+ p^2\lambda_e^{-1} J_1(\lambda_e) \{(x_e - x)\beta - (y_e - y)\alpha\} \vartheta(\zeta;\ p)] d\xi\Big\}$$

$$\lambda_e = ip\,[(x_e - x)^2 + (y_e - y)^2]^{1/2}, \qquad w = u + iv, \quad \zeta = u + i\xi \tag{1.10}$$

In obtaining (1.10) we made use of the Cauchy-Riemann conditions for $v = 0$,

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} = \alpha(u), \quad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u} = -\beta(u) \qquad v = 0$$

the subscript "e" in (1.10) arises because of the fact that integration is carried out along that half of AB where $v = 0$; x_e, y_e, α, β are functions of ζ; the subscripts "c" are omitted, so that the observation point now has the coordinates x, y; u, v.

In constructing the solution in the form of Fourier series it is helpful to carry out the following transformation (identical in the case of infinite series):

$$\varphi(x, y, z) = \operatorname{Re}\Big\{V(w, z) + \int_0^v F(\zeta, z)\, d\xi + \int_0^v \Big[\sum_{p=0}^{\infty} \{J_0(\lambda_e) - 1\} \langle f e^{ipz}\rangle +$$

$$+ \{(x_e - x)\beta - (y_e - y)\alpha\} p^2 \lambda_e^{-1} J_1(\lambda_e) \langle \vartheta e^{ipz}\rangle\Big] d\xi\Big\}$$

$$\langle f e^{ipz}\rangle = f_p^{(c)} \cos pz + f_p^{(s)} \sin pz \tag{1.11}$$

This expression satisfies the conditions at the boubdary exactly with any number of terms retained in the series; $f_p^{(c)}$ and $f_p^{(s)}$ are the Fourier coefficients of the function F.

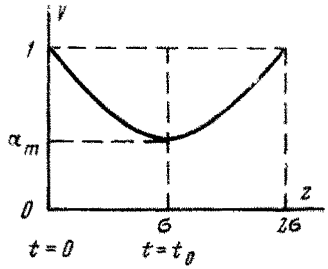

Fig. 3

1.2. Examples: approximate and exact solutions for periodic focusing. In considering the problem of periodic focusing of plane and cylindrical beams the author of [5] adopted the following approximation of the true potential distribution at the boundary (Fig. 3):

$$V = 1 - (1 - \alpha_m)\cos(\pi z / 2\sigma), \qquad F = 0 \tag{1.12}$$

Let us cut the domain Ω out of a plane diode in which the potential varies as shown in Fig. 3. Making use of the above formulas and recalling that $\vartheta_0 = 1$, $\vartheta_1 = \alpha_m - 1$, $\vartheta_p = 0$ $(p = 2, 3, \ldots)$, $f_p = 0$ $(p = 0, 1, \ldots)$ we obtain an approximate solution for periodic focusing of a cylindrical beam of arbitrary cross section in the form

$$\varphi(x, y, z) = 1 - (1 - \alpha_m)\left\{1 - \left(\frac{\pi}{2\sigma}\right)^2 \operatorname{Re}\int_0^v \lambda_e^{-1} J_1(\lambda_e)[(x_e - x)\beta - (y_e - y)\alpha]\, d\xi\right\} \times$$

$$\times \cos \frac{\pi z}{2\sigma}, \quad p = \frac{\pi}{2\sigma} \tag{1.13}$$

The exact expression for $V = V(z)$ can be given parametrically, (1.14)

$$V = (1/_9 t^2 - 2/_9 t_0 t + 1)^2, \quad z = 1/_{27} t^3 - 1/_9 t_0 t^2 + t, \quad 0 \leqslant t \leqslant t_0, \quad t_0 = 3\sqrt{1 - \sqrt{\alpha_m}}$$

It is sufficient to determine the potential in the interval $0 \leqslant z \leqslant \sigma$. The Fourier coefficients for (1.14) then become

$$\vartheta_p^{(c)} = \frac{4}{3p^2} \frac{1}{1 + 2\sqrt{\alpha_m}} - \frac{4}{9} \frac{1}{\sigma p^2} \int_0^{t_0} \cos pz \, dt, \quad \vartheta_p^{(s)} = -\frac{1 - \alpha_m}{\sigma p} + \frac{4}{9} \frac{1}{\sigma p^2} \int_0^{t_0} \sin pz \, dt$$

$$(p = 2\pi q / \sigma, q = 1, 2, \ldots)$$

and formula (1.11) yields

$$\varphi(x, y, z) = V(z) + \operatorname{Re} \sum_{p=1}^{\infty} p^2 \langle \vartheta e^{ipz} \rangle \int_0^{v} \lambda_e^{-1} J_1(\lambda_e)[(x_e - x)\beta - (y_e - y)\alpha] \, d\xi \tag{1.15}$$

2. An elliptical cylinder. It is possible in certain cases to express the solution in more compact form by carrying out analysis in a system attached to the beam surface. Let ξ, η, z be such a system and let $\xi = \xi_0$ be the equation of Σ .

$$\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} + \sqrt{g} \frac{\partial \varphi}{\partial z^2} = 0$$

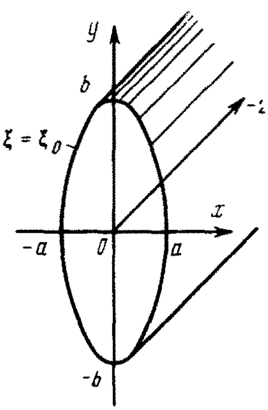

Fig. 4

Introducing the new variables

$$\zeta = \xi + i\eta, \quad \bar{\zeta} = \xi - i\eta$$

we find that the two-dimensional potential must satisfy the equation

$$4 \frac{\partial^2 \Phi}{\partial \zeta \partial \bar{\zeta}} = p^2 \sqrt{g} \Phi \tag{2.1}$$

If $\sqrt{g} = a'(\zeta) b'(\bar{\zeta})$, then the introduction of λ by way of the formula

$$\lambda = ip \{[a(\zeta) - a(\zeta_c)]\} \{[b(\bar{\zeta}) - b(\bar{\zeta}_c)]\}^{1/2}$$

yields the zero-order Bessel function $J_0(\lambda)$ for G Separation of variables in the determinant of the metric tensor is possible, for example, in elliptical and parabolic coordinates. For an elliptical cylinder (Fig. 4) we have

$$x = a\sqrt{\beta - 1} \operatorname{sh} \xi \sin \eta, \quad y = a\sqrt{\beta - 1} \operatorname{ch} \xi \cos \eta, \quad \operatorname{th}^2 \xi_0 = 1 / \beta$$

$$a(\zeta) = a\sqrt{\beta - 1} \operatorname{ch} \zeta, \quad b(\bar{\zeta}) = a\sqrt{\beta - 1} \operatorname{ch} \bar{\zeta}, \quad \beta = (b / a)^2$$

Making use of the formula given in [10] and carrying out the appropriate transformations, we arrive at the following expression for $\Phi(\xi, \eta; p)$:

$$\Phi(\xi, \eta; p) = \operatorname{Re} \vartheta(T; p) + \operatorname{Im} \int_0^T \{I_0(k_e) f(\tau; p) -$$

$$- pa\sqrt{\beta - 1} I_1(k_e) \operatorname{Re} [\operatorname{sh}(\xi_0 - i\tau) e^{i\psi}] \vartheta(\tau; p)\} d\tau$$

$$T = \eta + i(\xi - \xi_0), \quad k_e = pa\sqrt{\beta - 1} \rho \tag{2.2}$$

$$\rho = |\operatorname{ch}(\xi_0 + i\tau) - \operatorname{ch}(\xi + i\eta)|, \quad \psi = \arg [\operatorname{ch}(\xi_0 + i\tau) - \operatorname{ch}(\xi + i\eta)]$$

In deriving (2.2) we took into account the fact that λ is a purely imaginary argument. In the case of an elliptical beam with potential (1.12) the use of general formula

(2.2) yields

$$\varphi(\xi,\eta,z)=1-(1-\alpha_m)\left\{1+\frac{\pi}{2\sigma}a\sqrt{\beta-1}\ \operatorname{Im}\int_0^T I_1(k_e)\times\right.$$
$$\left.\times\operatorname{Re}\left[\operatorname{sh}(\xi_0-i\tau)e^{i\psi}\right]d\tau\right\}\cos\frac{\pi z}{2\sigma} \tag{2.3}$$

Now let us consider the problem of shaping of a flow with an elliptical boundary in which the z-component of the velocity varies periodically [6]. The Cauchy conditions at $\xi=\xi_0$ are

$$V(\eta,z)=V_0(\eta)+W(z)=\sum_{p=0}^{\infty}\vartheta_p(\eta;p)\cos pz,\qquad F=F(\eta) \tag{2.4}$$

The function $W=W(z)$ can be expressed parametrically,

$$W=(1-\gamma\cos t)^2,\qquad z=t-\gamma\sin t$$

The functions $V_0(\eta)$ and $F(\eta)$ are explained in [3],

$$V_0(\eta)=\frac{\alpha}{2}\frac{\beta+1}{\beta}\left(1-\frac{\beta-1}{\beta+1}\cos 2\eta\right),\quad F(\eta)=\frac{\alpha(\beta^2+1)}{\beta^{3/2}}\left(1-\frac{\beta^2-1}{\beta^2+1}\cos 2\eta\right),\ \alpha=\mathrm{const}$$

The Fourier coefficients in conditions (2.4) are given by

$$\vartheta_0=V_0(\eta),\ \vartheta_p=\frac{1}{\pi}\int_0^{2\pi}(1-\gamma\cos t)^3\cos\left[p(t-\gamma\sin t)\right]dt=-\frac{4}{p^2}J_p(p\gamma)$$
$$f_0=F(\eta),\quad f_p=0\quad(p=1,2,\ldots) \tag{2.5}$$

Here J_p is a Bessel function. Making use of (2.2), (2.5), we arrive at the following expression for the potential:

$$\varphi(\xi,\eta,z)=\alpha\left[\frac{\beta+1}{2\beta}+\frac{\beta^2+1}{\beta^{3/2}}(\xi-\xi_0)-\frac{\beta-1}{2\beta}\cos 2\eta\left\{\operatorname{ch}2(\xi-\xi_0)+\frac{\beta+1}{\beta^{1/2}}\operatorname{sh}2(\xi-\xi_0)\right\}\right]+$$
$$+W(z)+4a\sqrt{\beta-1}\sum_{p=1}^{\infty}\left\{\frac{J_p(p\gamma)}{p}\cos pz\ \operatorname{Im}\int_0^T I_1(k_e)\operatorname{Re}\left[\operatorname{sh}(\xi_0-i\tau)e^{i\psi}\right]d\tau\right\} \tag{2.6}$$

3. Irregular Cauchy conditions for a cylindrical domain of arbitrary cross section. If the Cauchy conditions at Σ become irregular functions, then the Fourier series apparatus is no longer applicable anf must be replaced by Laplace transformation. Here each specific case requires special analysis, since the solving procedure entails regularization of definite integrals which depends on the form of the integrand. We shall consider the practically important case where the domain Ω is cut out of a plane diode with emission bounded by a space charge. We then have

$$V=z^{4/3},\qquad F=0 \tag{3.1}$$

at the surface Σ of the cylinder.

We can show [1] that $z^{4/3}$ is given by

$$z^{4/3}=\frac{1}{\Gamma(-4/3)}\int_{(0)}^{\infty}\frac{e^{-pz}}{p^{7/3}}dp \tag{3.2}$$

Here $\Gamma(x)$ is a gamma function. The Cauchy conditions for $\Phi(u,\ v;\ p)$ are as follows:

$$\vartheta=\vartheta(p)=p^{-7/3}/\Gamma(-4/3),\qquad f=0 \tag{3.3}$$

Substituting (3.3) into (1.10) and replacing p by ip in this formula, we obtain the solution in the form

$$\varphi(x, y, z) = \frac{1}{\Gamma(-4/3)} \int_{(0)}^{\infty} \frac{e^{-pz}}{p^{7/3}} \Big\{1 + p^2 \operatorname{Re} \int_0^v \frac{J_1(\lambda_e)}{\lambda_e} [(x_e - x)\beta - (y_e - y)\alpha] \, d\xi\Big\} dp \tag{3.4}$$

$$\lambda_e = -p[(x_e - x)^2 + (y_e - y)^2]^{1/2}, \; x_e = x_e(\zeta), \; y_e = y_e(\zeta), \alpha = \alpha(\zeta)$$

$$\beta = \beta(\zeta), \; \zeta = u + i\xi$$

Contour integration in the p-plane in (3.4) can be carried out with the aid of the Lipschitz-Hankel integral

$$\int_{(0)}^{\infty} e^{-ap} p^{q-1} J_\nu(bp)\, dp = \frac{b^\nu \Gamma(\nu + q)}{2^\nu (a^2 + b^2)^{1/2 (\nu+q)} \Gamma(1+\nu)} F\Big(\frac{\nu+q}{2}, \frac{1+\nu-q}{2}, \nu+1; \frac{b^2}{a^2+b^2}\Big) \tag{3.5}$$

$$\operatorname{Re}[(a \pm ib)p] > 0, \quad p \to \infty$$

Here $F(a, b, c; z)$ is a hypergeometric function. Thus,

$$\varphi(x, y, z) = z^{4/3} - \frac{1}{\Gamma(-4/3)} \operatorname{Re} \int_0^v \Big[\frac{(x_e - x)\beta - (y_e - y)\alpha}{r} \int_{(0)}^{\infty} \frac{e^{-pz}}{p^{4/3}} J_1(-pr)\, dp\Big] d\xi =$$

$$= z^{4/3} + \frac{\Gamma(2/3)}{2\Gamma(-4/3)} \operatorname{Re} \int_0^v \frac{(x_e - x)\beta - (y_e - y)\alpha}{(z^2 + r^2)^{1/3}} F\Big(\frac{1}{3}, \frac{7}{6}, 2; \frac{r^2}{z^2 + r^2}\Big) d\xi \tag{3.6}$$

where

$$r = [(x_e - x)^2 + (y_e - y)^2]^{1/2}$$

can be regarded as the analytical continuation of the distance between the projection of the point of observation of the plane x, y and a point on Γ. On fulfillment of the additional condition

$$|a + ib| \leqslant |a - ib| \tag{3.7}$$

the hypergeometric function in (3.5) can be expressed in terms of the Legendre function. Since $z \geqslant 0$, we must replace condition (3.7) by the requirement that $\operatorname{Im} r \leqslant 0$. Thus, for $\operatorname{Im} r \leqslant 0$ we have

$$\varphi(x, y, z) = z^{4/3} - \operatorname{Re} \int_0^v \frac{(x_e - x)\beta - (y_e - y)\alpha}{[z^2 + (x_e - x)^2 + (y_e - y)^2]^{1/3}} \frac{dP_{-4/3}(\Theta)}{d\Theta} d\xi \tag{3.8}$$

$$\Theta = [1 + (r/z)^2]^{-1/2}, \; \operatorname{Im} r \leqslant 0$$

Expressions (3.6), (3.8) yield the solution of our problem.

3.1. The angle of inclination of the shaping electrode at zero potential. Let us consider expression (3.7) in the neighborhood of emitting surface $z = 0$ and near the beam boundary Σ. The smallness of v, ξ, z enables us to make certain simplifications,

$$x_e(\zeta) = x_e(u + i\xi) = x_e(u) + \alpha(u) i\xi$$

$$y_e(\zeta) = y_e(u + i\xi) = y_e(u) + \beta(u) i\xi$$

$$x + iy = x_e(u + iv) + iy_e(u + iv) = x(u, v) + iy(u, v)] \tag{3.9}$$

$$x(u, v) = x(u, 0) + \Big(\frac{\partial x}{\partial v}\Big)_{v=0} v = x_e(u) - \beta(u) v$$

$$y(u, v) = y(u, 0) + \Big(\frac{\partial y}{\partial v}\Big)_{v=0} v = y_e(u) + \alpha(u) v$$

We have made use of the aforementioned Cauchy-Riemann conditions for $v = 0$ in the two latter equations. Applying (3. 9), we obtain

$$r^2 = (\alpha^2 + \beta^2)(v^2 - \xi^2), \qquad (x_e - x)\beta - (y_e - y)\alpha = (\alpha^2 + \beta^2)v$$

The equation of the zero equipotential with allowance for the above formulas becomes

$$0 = z^{4/3} - z^{-2/3}(\alpha^2 + \beta^2)v\int_0^v \left[1 + (\alpha^2 + \beta^2)\frac{v^2}{z^2}\left(1 - \frac{\xi^2}{v^2}\right)\right]^{-1/3} \times$$
$$\times P_{-4/3}^{1\prime}\left(\left[1 + (\alpha^2 + \beta^2)\frac{v^2}{z^2}\left(1 - \frac{\xi^2}{v^2}\right)\right]^{-1/2}\right)d\xi$$

where the prime denotes the derivative with repect to the argument. Recalling that the distance measured along the normal to Γ is related to v by the expression

$$(\alpha^2 + \beta^2)v^2 = n^2$$

introducing the new integration variable $\eta = \xi / v$, and making use of the explicit expression of the zero equipotential in the plane n, z

$$n = \theta z + \ldots, \qquad \theta = \theta(u)$$

we obtain the following equation for determining θ:

$$\frac{1}{\theta^2} = \int_0^1 [1 + \theta^2(1 - \eta^2)]^{-1/3} P_{-4/3}'([1 + \theta^2(1 - \eta^2)]^{-1/2})\,d\eta \tag{3.10}$$

Equation (3. 10) indicates that the inclination of the zero equipotential does not depend on the shape of the boundary. We know that in the case of plane and axisymmetric flows (and specifically in the case of an ordinary cylindrical beam) this angle is equal to 67°. 5. Thus, the θ in (3. 9) is

$$\theta = \text{const} = 1 + \sqrt{2} \tag{3.11}$$

Computation of the coefficients in the (n, z)-equation of the zero equipotential can be continued. The resulting definite integrals, which do not depend on the shape of the boundary, can be estimated from the known expansion for a circular cylindrical beam obtainable from the formulas of [11]. Thus, the curvature of the zero shaping electrode turns out to be proportional to the curvature k_Γ of the contour Γ, where the proportionality coefficient is the numerical value of the curvature of the zero equipotential in the case of a circular cylinder

$$k_\varphi = \frac{9}{28}\sin\frac{\pi}{8}\frac{\alpha\beta' - \alpha'\beta}{(\alpha^2 + \beta^2)^{3/2}} = \frac{9}{28}\sin\frac{\pi}{8}k_\Gamma$$

3. 2. Cylindrical and elliptical beams. For a cylindrical beam with a disk-shaped cross section we have

$$x = x_e(t) = \cos t, \quad y = y_e(t) = \sin t, \quad \alpha(t) = -\sin t, \quad \beta(t) = \cos t$$
$$x + iy = \text{Re}^{i\psi} = e^{iw}, \quad x = e^{-v}\cos u, \quad y = e^{-v}\sin u, \; u = \psi, \quad v = -\ln R$$
$$(x_e - x)\beta - (y_e - y)\alpha = 1 - e^{-v}\,\text{ch}\,\xi$$
$$r^2 = (x_e - x)^2 + (y_e - y)^2 = 2e^{-v}(\text{ch}\,v - \text{ch}\,\xi)$$

Since r is real, we can make use of formula (3. 8). The appropriate transformations yield

$$\varphi(R, z) = z^{4/3} + z^{-2/3}\int_1^R \frac{1}{\eta}\left[1 - \frac{R}{2}\left(\eta + \frac{1}{\eta}\right)\right][\Theta(R, \eta)]^{2/3} P_{-4/3}(\Theta)\,d\eta$$

$$\Theta(R, \eta) = \left[1 + \frac{R}{z^2}\left(R - \eta + \frac{1}{R} - \frac{1}{\eta}\right)\right]^{-1/2} \tag{3.12}$$

Expression (3.12) is a new form of the solution different from that given in [2].

It is clear that direct application of the procedure of Sect. 1 to an elliptical beam yields extremely cumbersome expressions for the integrand. Let us therefore use the results of Sect. 2. Condition (3.7) is always fulfilled, since a and b are real in this case. This enables us to write

$$\varphi(\xi, \eta, z) = z^{4/3} - a^2(\beta - 1) z^{-2/3} \operatorname{Im} \int_0^T \rho \operatorname{Re}\left[\operatorname{sh}(\xi_0 - i\tau) e^{i\psi}\right] Z^{2/3} P'_{-4/3}(Z)\, d\tau$$

$$T = \eta + i(\xi - \xi_0), \quad Z = \left[1 + a^2(\beta - 1)(\rho / z)^2\right]^{-1/2} \tag{3.13}$$

$$\operatorname{ch}(\xi_0 + i\tau) - \operatorname{ch}(\xi + i\eta) = \rho e^{i\psi}$$

3.3. Some generalizations. It is clear that the results expressed by formulas (3.5), (3.8) can be readily extended to the case where the boundary conditions are given by arbitrary power functions of z with coefficients dependent on u

$$V(u, z) = V_0(u) z^{\nu}, \qquad F(u, z) = F_0(u) z^{\mu} \tag{3.14}$$

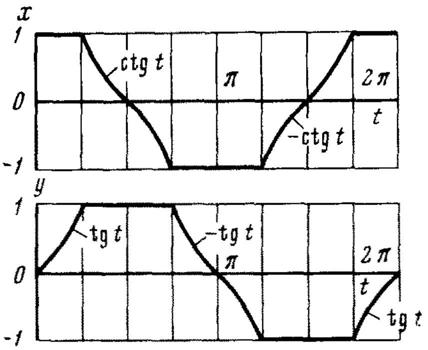

Fig. 5

Conditions of the type (3.14) occur in computing a slightly curved cylindrical beam.

In constructing the solution we assumed that the functions $x_e(t)$, $y_e(t)$ are analytic. However, analytic parametric functions can describe curves of the astroid type which contain cusps. The zero equipotential then contains lines of discontinuity. At the same time, if we have an exact solution which can be used for the analytic parametric specification of Γ, then we can attempt to construct an approximate solution for nonanalytic parametric relations by expanding them in a Fourier series. For example, if Γ is a square, then it is described by the functions $x_e(t)$, $y_e(t)$ shown in Fig. 5. Here

$$x_e(t) = \sum_{k=0}^{\infty} a_{2k+1} \cos(2k + 1) t, \qquad y_e(t) = \sum_{k=0}^{\infty} b_{2k+1} \sin(2k + 1) t$$

$$a_{4q+1} = a_{4q-1} = b_{4q+1} = -b_{4q-1} = -\frac{4}{\pi}\left[2 \sum_{l=1}^{2q} \frac{\cos {}^1\!/_4 (2l - 1)\pi}{2l - 1} + \ln \operatorname{tg} \frac{\pi}{8}\right]$$

Here t is an ordinary polar angle; the pole lies at the center of the square whose side is equal to 2.

4. Domains with axial symmetry. In the axisymmetric case we use the Laplace equation written out in the ordinary cylindrical coordinates R, ψ, z

$$\frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial \varphi}{\partial R}\right) + \frac{1}{R^2} \frac{\partial^2 \varphi}{\partial \psi^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{4.1}$$

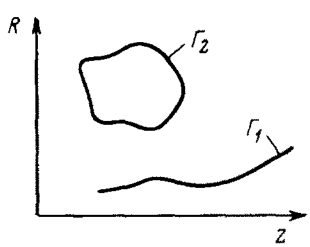

Fig. 6

The domain Ω is obtained by rotating the curves Γ_1, Γ_2 shown in Fig. 6 about the z-axis; Γ_1 yields the surface of revolution with an arbitrary law of variation of the cross-section radius; Γ_2 yields a toroidal domain of arbitrary cross section. The curves Γ are given by the parametric equations

$$R = R_e(t), \quad z = z_e(t); \; dR_e \,/\, dt = \alpha\,(t), \quad dz_e \,/\, dt = \beta\,(t) \tag{4.2}$$

and the Cauchy conditions for φ by the formulas

$$\varphi\,|_\Sigma = V_e\,(t,\,\psi), \qquad \partial\varphi \,/\, \partial n\,|_\Sigma = V_{ne}\,(t,\,\psi) \tag{4.3}$$

4.1. Solving the two-dimensional problem. The problem consists in determining the two-dimensional potential $\Phi\;(R,\; z;\; p)$,

$$\frac{1}{R}\,\frac{\partial}{\partial R}\left(R\,\frac{\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial z^2} - \frac{p^2}{R^2}\,\Phi = 0 \tag{4.4}$$

satisfying the conditions

$$\Phi\,|_{v=0} = \vartheta\,(u;\,p) \qquad \partial\Phi \,/\, \partial v\,|_{v=0} = f\,(u;\,p) \tag{4.5}$$

on Γ.

The mapping of the ancillary plane $w = u + iv$ onto the meridian plane is defined by the function

$$z + iR = z_e\,(w) + iR_e\,(w) \tag{4.6}$$

Solution of problem (4.4), (4.5) consists largely in repeating the argument of [2], the only difference being that the Riemann function is now the hypergeometric function

$$G = F\left(\frac{1}{2} + p,\, \frac{1}{2} - p,\, 1;\, \lambda\right), \quad \lambda = -\,(4RR_c)^{-1}\,[(R - R_c)^2 + (z - z_c)^2] \tag{4.7}$$

Omitting the intervening discussion, we simply write out the solution in its final form,

$$\begin{aligned}\Phi\,(u,\,v;\,p) = \mathrm{Re}\Bigg\{\left[\frac{R_e\,(w)}{R}\right]^{1/2} \vartheta\,(w;\,p) + \int\limits_0^v \Bigg[\left(\frac{R_e}{R}\right)^{1/2}\left(f + \frac{\beta\vartheta}{2R_e}\right) F\left(\frac{1}{2} + p,\right.\\ \left.\frac{1}{2} - p,\, 1;\, \lambda_e\right) - \left(\frac{1}{4} - p^2\right)\frac{\vartheta}{2R_e^{1/2}R^{3/2}}\left(\frac{R^2 - R_e^2 + [z_e - z]^2}{2R_e}\,\beta + [z_e - z]\,\alpha\right) \times \\ \times F\left(\frac{3}{2} + p,\, \frac{3}{2} - p,\, 2;\, \lambda_e\right)\Bigg]\,d\xi\Bigg\}\end{aligned} \tag{4.8}$$

$$\lambda_e = (4R_eR)^{-1}\;[(R_e - R)^2 + (z_e - z)^2], \quad w = u + iv, \quad \zeta = u + i\xi$$

The symbols R_e, z_e, α, β, ϑ, f in the integrand are functions of ζ; R and z are functions of u, v, which must be determined from (4.6).

If Cauchy conditions (4.3) are regular in ψ, then the solution of the initial three-dimensional problem can be written as

$$\begin{aligned}\varphi\,(R,\,\psi,\,z) = \mathrm{Re}\Bigg\{\left[\frac{R_e\,(w)}{R}\right]^{1/2} V\,(w,\,\psi) + \int\limits_0^v \left[\frac{R_e\,(\zeta)}{R}\right]^{1/2}\left[F\,(\zeta,\,\psi) + \frac{\beta}{2R_e}\,V\,(\zeta,\,\psi)\right] d\xi\, + \\ + \int\limits_0^v \Bigg[\left(\frac{R_e}{R}\right)^{1/2} \sum_{p=0}^{\infty}\left\{F\left(\frac{1}{2} + p,\, \frac{1}{2} - p,\, 1;\, \lambda_e\right) - 1\right\}\left(\langle f e^{ip\psi}\rangle + \frac{\beta}{2R_e}\,\langle\vartheta e^{ip\psi}\rangle\right) - \\ - \frac{1}{2R_e^{1/2}\,R^{3/2}}\left(\frac{R^2 - R_e^2 + [z_e - z]^2}{2R_e}\,\beta + [z_e - z]\,\alpha\right)\sum_{p=0}^{\infty}\left(\frac{1}{4} - p^2\right) \times \\ \times F\left(\frac{3}{2} + p,\, \frac{3}{2} - p,\, 2;\, \lambda_e\right)\langle\vartheta e^{ip\psi}\rangle\Bigg]\,d\xi\Bigg\},\quad \langle\vartheta e^{ip\psi}\rangle = \vartheta_p^{(c)}\cos p\psi + \vartheta_p^{(s)}\sin p\psi\end{aligned} \tag{4.9}$$

Here $\vartheta_p^{(c)}$, $\vartheta_p^{(s)}$ are the Fourier coefficients of the function $V\,(u,\,\psi)$; the R_e, z_e, α, β,

ϑ, f in the integrand are functions of ζ. Expressions (4. 9) satisfy the boundary conditions exactly for any number of terms of the series in p. We assume that V and F are periodic functions with the period 2π. Extension of the above results to the case of unequal periods different from 2π is trivial. Formula (4. 9) extends the solution of [2] to the case where the Cauchy conditions depend on the azimuth and also to the case of a toroid of arbitrary cross section.

5. A toroidal domain with an irregular function. Let us cut the domain Ω for the curve Γ_2 out of the flow with circular particle trajectories investigated in [7]. The Cauchy conditions are given by the formulas

$$\varphi\,|_{v=0} = V(t, \psi) = R_e^{-2}(t)\,(\sin {}^3/_2\,\psi)^{4/3}$$
$$\partial\varphi / \partial v\,|_{v=0} = \partial\varphi / \partial R\,|_\Sigma\, \beta = F(t, \psi) = -2\beta(t)\, R_e^{-3}(t)\,(\sin {}^3/_2\,\psi)^{4/3} \tag{5.1}$$

Making use of the Laplace transform

$$\left(\sin\frac{3}{2}\psi\right)^{4/3} = \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} \theta(p)\, e^{p\psi}\, dp, \qquad a > 0 \tag{5.2}$$

we obtain the following expressions for ϑ and f :

$$\vartheta(t;\, p) = R_e^{-2}(t)\,\theta(p), \qquad f(t;\, p) = -2\beta(t) R_e^{-3}(t)\theta(p) \tag{5.3}$$

All we need do now is find $\theta(p)$, make use of formula (4. 8), and express $\varphi(R, \psi, z)$ in the form (5. 2).

$$\theta(p) = \int\limits_0^\infty \left(\sin\frac{3}{2}\psi\right)^{4/3} e^{-p\psi}\, d\psi = \frac{2}{3}\,\frac{1 + e^{-\pi q}}{1 - e^{-2\pi q}}\int\limits_0^\pi \sin^{4/3}\psi\, e^{is\psi}\, d\psi$$
$$s = iq = {}^2/_3\, ip$$

The integral in the right side can be expressed in terms of the B-function,

$$\int\limits_0^\pi e^{i(p-q)\psi} \sin^{p+q-2}\psi\, d\psi = e^{{}^1/_2 i(p-q)\pi}\,[2^{p+q-2}(p+q-1)\,\mathrm{B}(p, q)]^{-1}$$

On carrying out the transformations we obtain

$$\theta(p) = \frac{\pi}{14\sqrt[3]{2}}\,\Gamma\left(\frac{10}{3}\right)\left[\mathrm{sh}\,\frac{\pi p}{3}\,\Gamma\left(\frac{5}{3} + \frac{1}{3}ip\right)\Gamma\left(\frac{5}{3} - \frac{1}{3}ip\right)\right]^{-1} \tag{5.4}$$

The unique singularity of the simple-pole type lies at the point $p = 0$. Following the above procedure, we obtain the two-dimensional potential

$$\Phi\;(\;, v;\, p) = \frac{\theta(p)}{\sqrt{R}}\,\mathrm{Re}\left\{[R_e(w)]^{-3/2} - \int\limits_0^v\left[\frac{3}{2}\,\beta F\left(\frac{1}{2} + ip, \frac{1}{2} - ip, 1;\, \lambda_e\right) + \right.\right.$$
$$+ \frac{1}{2R}\left(-\frac{1}{4} + p^2\right)\left(\frac{R^2 - R_e^2 + [z_e - z]^2}{2R_e}\,\beta + [z_e - z]\,\alpha\right) F\left(\frac{3}{2} + ip, \frac{3}{2} - ip, 2;\, \lambda_e\right)\Bigg] \times$$
$$\left.\times\, R_e^{-5/2}\, d\xi\right\} \tag{5.5}$$

Finally, making use of (5. 5) and recalling (5. 2), we obtain

$$\varphi(R, \psi, z) = \frac{(\sin {}^3/_2\,\psi)^{4/3}}{\sqrt{R}}\,\mathrm{Re}\,[R_e(w)]^{-3/2} +$$

$$+\frac{1}{\sqrt{R}}\operatorname{Re}\int_0^v\left\{-\frac{3}{2}\beta\left[\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\theta(p)F\left(\frac{1}{2}+ip,\frac{1}{2}-ip,1;\lambda_e\right)e^{p\psi}dp\right]+\right.$$
$$+\frac{1}{2R}\left(\frac{R^2-R_e^2+[z_e-z]^2}{2R_e}\beta+[z_e-z]\alpha\right)\left[\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{1}{4}+p^2\right)\theta(p)\times\right.$$
$$\left.\left.\times F\left(\frac{3}{2}+ip,\frac{3}{2}-ip,2;\lambda_e\right)e^{p\psi}\,dp\right]\right\}R_e^{-3/2}d\xi \qquad (5.6)$$

Estimation of the contour integrals in (5.6) constitutes an independent problem. We note that the potential at sufficient distances from the emitting and collecting surfaces can be computed with the aid of the Fourier series

$$\left(\sin\frac{3}{2}\psi\right)^{4/3}=\sum_{k=0}^{\infty}\theta_k\cos 3k\psi,\quad \theta_k=\frac{3(-1)^k}{7\sqrt[3]{2}}\Gamma\left(\frac{10}{3}\right)\left[\Gamma\left(\frac{5}{3}+k\right)\Gamma\left(\frac{5}{3}-k\right)\right]^{-1}\left(1-\frac{1}{2}\delta_{0k}\right) \qquad (5.7)$$

The potential is given by the expression

$$\varphi(R,\psi,z)=\frac{(\sin{}^3/_2\,\psi)^{4/3}}{\sqrt{R}}\operatorname{Re}\left\{[R_e(w)]^{-3/2}-\frac{3}{2}\int_0^v\beta(\zeta)\,[R_e(\zeta)]^{-3/2}\,d\xi\right\}-$$
$$-\operatorname{Re}\int_0^v\left\{\frac{3}{2}\beta\sum_{k=0}^{\infty}\theta_k\left[F\left(\frac{1}{2}+k,\frac{1}{2}-k,1;\lambda_e\right)-1\right]\cos 3k\psi+\right.$$
$$+\frac{1}{2R}\left(\frac{R^2-R_e^2+[z_e-z]^2}{2R_e}\beta+[z_e-z]\alpha\right)\sum_{k=0}^{\infty}\left(\frac{1}{4}-k^2\right)\theta_k\times$$
$$\left.\times F\left(\frac{3}{2}+k,\frac{3}{2}-k,2;\lambda_e\right)\cos 3k\psi\right\}R_e^{-5/2}\,d\xi \qquad (5.8)$$

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